

# Heptagonal Symmetry for Quarks and Leptons

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## Abstract

The non-Abelian discrete symmetry  $D_7$  of the heptagon is successfully applied to both quark and lepton mass matrices, including  $CP$  violation.

# 1 Introduction

The structure of quark and lepton mass matrices has been under theoretical study for many years. Whereas the 6 quark masses and the 3 mixing angles and 1  $CP$  violating phase in the quark sector are now measured with some precision, the lepton sector is still missing some crucial information. Recently, the neutrino mixing angle  $\theta_{13}$  has been measured by the Daya Bay [1] and RENO [2] collaborations. The fact that  $\sin^2 2\theta_{13}$  is now centered at around 0.1 means that the previously favored tribimaximal mixing pattern ( $\sin^2 \theta_{23} = 1/2$ ,  $\sin^2 \theta_{12} = 1/3$ ,  $\theta_{13} = 0$ ) is invalid, although the  $A_4$  symmetry [3, 4, 5] used to obtain it [6] is still applicable with some simple modifications [7, 8, 9]. On the other hand, in the simplest application [3, 5] of  $A_4$ , all the quark mixing angles are zero. The question is whether there exists another symmetry which successfully yields both quark and lepton mass matrices, with good fits of all masses, mixing angles, and phases. The answer is yes, as elaborated below.

Using the non-Abelian discrete symmetry  $D_7$  of the heptagon, it has been shown [10] that the  $CP$  violating phase of the quark mixing matrix may be predicted, whereas  $D_7$  also yields a pattern [11] for the neutrino mass matrix consistent with what is observed. This pattern is previously derived using the symmetry  $Q_8$  [12], and realizes a specific conjecture [13] that the neutrino mass matrix has two texture zeros in the basis that charged-lepton masses are diagonal.

In Sec. 2 the symmetry  $D_7$  is explained. In Sec. 3 the assignments of quarks under  $D_7$  are given with the accompanying Higgs structure and the resulting mass matrices. In Sec. 4 numerical fits to the quark masses and mixing angles are given, with a prediction of the  $CP$  violating phase. In Sec. 5 the assignments of leptons under  $D_7$  are given with the accompanying Higgs structure and the resulting mass matrices. In Sec. 6 the neutrino mass

matrix is analyzed to show that it allows for nonzero  $\theta_{13}$  and a specific correlation between it and  $\theta_{23}$  as well as  $\delta_{CP}$ . Given that  $\theta_{12}$  is close to the tribimaximal value, it prefers an inverted hierarchy of neutrino masses although a quasidegenerate pattern with either normal or inverted ordering cannot be ruled out. In Sec. 7 there are some concluding remarks.

## 2 Heptagonal Symmetry $D_7$

The group  $D_7$  is the symmetry group of the regular heptagon with 14 elements, 5 equivalence classes, and 5 irreducible representations. Its character table is shown below.

class	$n$	$h$	$\chi_1$	$\chi_2$	$\chi_3$	$\chi_4$	$\chi_5$
$C_1$	1	1	1	1	2	2	2
$C_2$	7	2	-1	1	0	0	0
$C_3$	2	7	1	1	$a_1$	$a_2$	$a_3$
$C_4$	2	7	1	1	$a_2$	$a_3$	$a_1$
$C_5$	2	7	1	1	$a_3$	$a_1$	$a_2$

Table 1: Character Table of  $D_7$ .

Here  $n$  is the number of elements and  $h$  is the order of each element. The numbers  $a_k$  are given by  $a_k = 2 \cos(2k\pi/7)$ . The character of each representation is its trace and must satisfy the following two orthogonality conditions:

$$\sum_{C_i} n_i \chi_{ai} \chi_{bi}^* = n \delta_{ab}, \quad \sum_{\chi_a} n_i \chi_{ai} \chi_{aj}^* = n \delta_{ij}, \quad (1)$$

where  $n = \sum_i n_i$  is the total number of elements. The number of irreducible representations must be equal to the number of equivalence classes.

The three irreducible two-dimensional representations of  $D_7$  may be chosen as follows. For

$\mathbf{2}_1$ , let

$$\begin{aligned} C_1 &: \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_2 : \begin{pmatrix} 0 & \omega^k \\ \omega^{7-k} & 0 \end{pmatrix}, \quad (k = 0, 1, 2, 3, 4, 5, 6), \\ C_3 &: \begin{pmatrix} \omega & 0 \\ 0 & \omega^6 \end{pmatrix}, \begin{pmatrix} \omega^6 & 0 \\ 0 & \omega \end{pmatrix}, \quad C_4 : \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^5 \end{pmatrix}, \begin{pmatrix} \omega^5 & 0 \\ 0 & \omega^2 \end{pmatrix}, \\ C_5 &: \begin{pmatrix} \omega^4 & 0 \\ 0 & \omega^3 \end{pmatrix}, \begin{pmatrix} \omega^3 & 0 \\ 0 & \omega^4 \end{pmatrix}, \end{aligned} \quad (2)$$

where  $\omega = \exp(2\pi i/7)$ , then  $\mathbf{2}_{2,3}$  are simply obtained by the cyclic permutation of  $C_{3,4,5}$ .

For  $D_n$  with  $n$  prime, there are  $2n$  elements divided into  $(n+3)/2$  equivalence classes:  $C_1$  contains just the identity,  $C_2$  has the  $n$  reflections,  $C_k$  from  $k = 3$  to  $(n+3)/2$  has 2 elements each of order  $n$ . There are 2 one-dimensional representations and  $(n-1)/2$  two-dimensional ones.

The group multiplication rules of  $D_7$  are:

$$\mathbf{1}' \times \mathbf{1}' = \mathbf{1}, \quad \mathbf{1}' \times \mathbf{2}_i = \mathbf{2}_i, \quad (3)$$

$$\mathbf{2}_i \times \mathbf{2}_i = \mathbf{1} + \mathbf{1}' + \mathbf{2}_{i+1}, \quad \mathbf{2}_i \times \mathbf{2}_{i+1} = \mathbf{2}_i + \mathbf{2}_{i+2}, \quad (4)$$

where  $\mathbf{2}_{4,5}$  means  $\mathbf{2}_{1,2}$ . In particular, let  $(a_1, a_2), (b_1, b_2) \sim \mathbf{2}_1$ , then

$$a_1 b_2 + a_2 b_1 \sim \mathbf{1}, \quad a_1 b_2 - a_2 b_1 \sim \mathbf{1}', \quad (a_1 b_1, a_2 b_2) \sim \mathbf{2}_2. \quad (5)$$

In the decomposition of  $\mathbf{2}_1 \times \mathbf{2}_2$ , we have instead

$$(a_2 b_1, a_1 b_2) \sim \mathbf{2}_1, \quad (a_2 b_2, a_1 b_1) \sim \mathbf{2}_3. \quad (6)$$

### 3 Quark Sector

We assign quarks as shown in Table 2 and Higgs doublets as shown in Table 3, together with an extra  $Z_2^d \times Z_2^u$  symmetry.

symmetry	$[(u, d), (c, s)]$	$(t, b)$	$(d^c, s^c)$	$b^c$	$(u^c, c^c)$	$t^c$
$D_7$	$\mathbf{2}_1$	$\mathbf{1}$	$\mathbf{2}_1$	$\mathbf{1}$	$\mathbf{2}_2$	$\mathbf{1}$
$Z_2^d$	+	+	−	−	+	+
$Z_2^u$	+	+	+	+	−	+

Table 2: Quark assignments under  $D_7 \times Z_2^d \times Z_2^u$ .

symmetry	$\Phi_1$	$\Phi_2$	$\Phi_{3,4}$	$\Phi_{5,6}$	$\Phi_{7,8}$
$D_7$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}_1$	$\mathbf{2}_2$	$\mathbf{2}_3$
$Z_2^d$	+	−	−	+	+
$Z_2^u$	+	+	+	+	−

Table 3: Higgs doublet assignments under  $D_7 \times Z_2^d \times Z_2^u$ .

As a result, the  $(u, c, t)$  mass matrix is diagonal, coming from the Yukawa terms  $uu^c\phi_7^0 + cc^c\phi_8^0$  and  $tt^c\phi_1^0$ . As for the  $(d, s, b)$  mass matrix, the allowed Yukawa terms are  $(ds^c + sd^c)\bar{\phi}_2^0$ ,  $bb^c\bar{\phi}_2^0$ ,  $b(d^c\bar{\phi}_4^0 + s^c\bar{\phi}_3^0)$ , and  $(d\bar{\phi}_4^0 + s\bar{\phi}_3^0)b^c$ . The resulting mass matrix is thus of the form [10]

$$\mathcal{M}_d = \begin{pmatrix} 0 & a & \xi b \\ a & 0 & b \\ \xi c & c & d \end{pmatrix}, \quad (7)$$

where  $\xi = \langle \bar{\phi}_4^0 \rangle / \langle \bar{\phi}_3^0 \rangle$ .

## 4 Prediction of $CP$ Phase

As in Ref. [10], we can redefine the phases of  $\mathcal{M}_d$  so that  $a, b, c, d$  are real, but  $\xi$  is complex.

Since  $\mathcal{M}_u$  is diagonal, we have

$$V_L^\dagger \mathcal{M}_d V_R = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}, \quad V_L^\dagger \mathcal{M}_d \mathcal{M}_d^\dagger V_L = \begin{pmatrix} m_d^2 & 0 & 0 \\ 0 & m_s^2 & 0 \\ 0 & 0 & m_b^2 \end{pmatrix}, \quad (8)$$

where  $V_L$  is the observed quark mixing matrix up to phase conventions. The structure of  $\mathcal{M}_d\mathcal{M}_d^\dagger$  allows us to obtain the following first approximations:

$$m_b \simeq \sqrt{c^2 + d^2}, \quad V_{cb} \simeq \frac{bd + \xi^* ac}{(1 + |\xi|^2)c^2 + d^2}, \quad V_{ub} \simeq \frac{ac + \xi bd}{c^2 + d^2}, \quad (9)$$

where  $a^2 \ll b^2$  and  $|\xi|^2 \ll 1$  are assumed. We now rotate  $\mathcal{M}_d\mathcal{M}_d^\dagger$  using

$$V_3 = \begin{pmatrix} 1 & 0 & V_{ub} \\ 0 & 1 & V_{cb} \\ -V_{ub}^* & -V_{cb}^* & 1 \end{pmatrix} \quad (10)$$

to obtain the  $2 \times 2$  matrix

$$\mathcal{M}_2\mathcal{M}_2^\dagger = \begin{pmatrix} A & C \\ C^* & B \end{pmatrix}, \quad (11)$$

where

$$A = a^2 + |\xi|^2 b^2 - |V_{ub}|^2 m_b^2, \quad (12)$$

$$B = a^2 + b^2 - |V_{cb}|^2 m_b^2, \quad (13)$$

$$C = \xi b^2 - V_{ub} V_{cb}^* m_b^2, \quad (14)$$

yielding

$$m_s^2 = \frac{1}{2}(B + A) + \frac{1}{2}\sqrt{(B - A)^2 + 4|C|^2}, \quad (15)$$

$$|V_{us}|^2 = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4|C|^2}{(B - A)^2 + 4|C|^2}}, \quad (16)$$

where the phase of  $V_{us}$  is that of  $C$ , and

$$m_d = |2abc\xi - a^2 d|/m_s m_b. \quad (17)$$

Using  $|V_{us}| = 0.22534$ , we find  $|C|^2/(B - A)^2 = 0.05971$ , and  $m_s^2 \gg m_d^2$  implies  $A \simeq 0.05351B$ , hence  $m_s^2 \simeq 1.05349B$ . Using these formulas, the 6 parameters  $a, b, c, d, Re(\xi), Im(\xi)$  may then be determined and the  $CP$  violating parameter  $J$  is predicted.

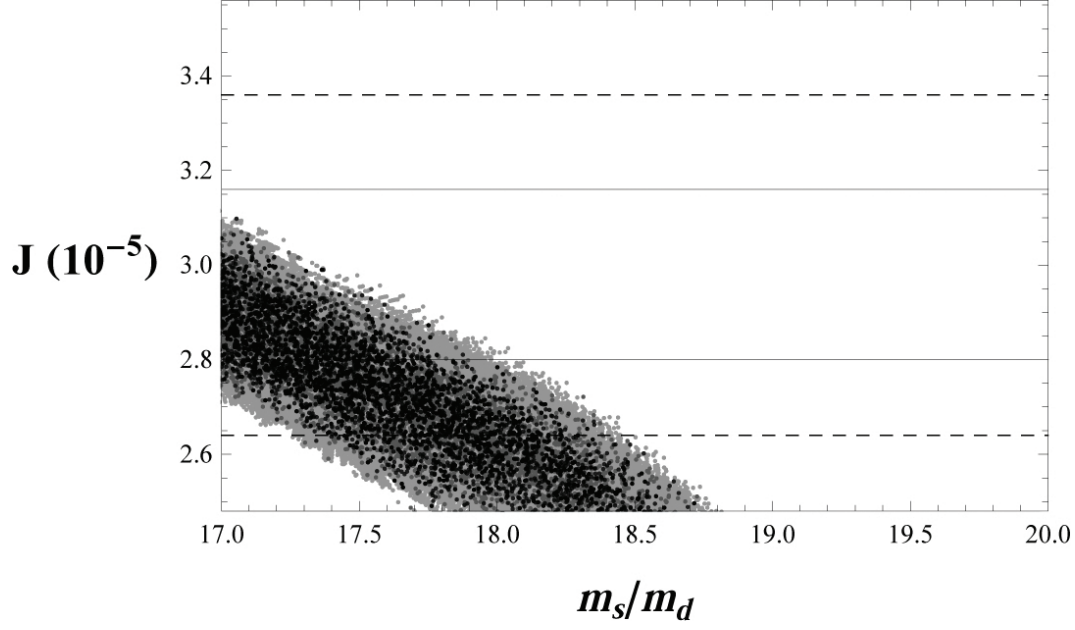


Figure 1: The  $CP$  violating parameter  $J$  versus  $m_s/m_d$ . The solid (dash) lines indicate the one (two) standard-deviation bounds of  $J$ .

For our numerical analysis, we dispense with the approximate expressions and diagonalize  $\mathcal{M}_d \mathcal{M}_d^\dagger$  directly. We scan for solutions consistent with data on the 3 masses and 3 mixing angles, within one standard deviation of each parameter. We then obtain  $J$  numerically from the resulting  $V_{CKM}$ . This is then the prediction of our model. In Fig. 1 we plot  $J$  versus  $m_s/m_d$ , which shows good agreement with data. We use the 2008 updated values [14] of  $m_{d,s,b}$  evaluated at  $M_W$ :

$$m_d(M_W) = 2.93 (+1.25/-1.21) \text{ MeV}, \quad (18)$$

$$m_s(M_W) = 56 \pm 16 \text{ MeV}, \quad (19)$$

$$m_b(M_W) = 2.92 \pm 0.09 \text{ GeV}, \quad (20)$$

and the 2012 Particle Data Group (PDG) [15] values of the mixing angles

$$|V_{us}| = 0.22534 \pm 0.00065, \quad (21)$$

$$|V_{cb}| = 0.0412 (+0.0011/-0.0005), \quad (22)$$

$$|V_{ub}| = 0.00351 (+0.00015/-0.00014). \quad (23)$$

Note that PDG also lists the condition  $17 < m_s/m_d < 22$  and the value of the  $CP$  violating parameter is

$$J = 2.96 (+0.20/-0.16) \times 10^{-5}. \quad (24)$$

We show in Table 4 sample values of  $a, b, c, d, Re(\xi), Im(\xi)$  with the corresponding values of  $m_d, m_s, m_b, |V_{us}|, |V_{ub}|, |V_{cb}|$  and  $J$  as well as  $m_s/m_d$ .

$a$ (GeV)	$b$ (GeV)	$c$ (GeV)	$d$ (GeV)	$Re(\xi)$	$Im(\xi)$	$m_s/m_d$
$m_d$ (MeV)	$m_s$ (MeV)	$m_b$ (GeV)	$ V_{us} $	$ V_{ub} $	$ V_{cb} $	J
0.0125	0.138	1.32	-2.60	0.053	-0.084	17.00
3.89	66.2	2.92	0.22534	0.00355	0.0420	$2.95 \times 10^{-5}$
0.0124	0.139	1.34	-2.60	0.058	-0.084	17.25
3.91	67.4	2.93	0.22532	0.00358	0.0420	$2.89 \times 10^{-5}$
0.0123	0.138	1.40	-2.60	0.064	-0.087	17.50
3.96	69.2	2.96	0.22519	0.00363	0.0409	$2.76 \times 10^{-5}$
0.0122	0.138	1.39	-2.55	0.068	-0.084	17.75
3.94	69.9	2.91	0.22501	0.00359	0.0415	$2.70 \times 10^{-5}$

Table 4:  $D_7$  parameter fits of quark masses and mixing.

## 5 Lepton Sector

Using again  $D_7 \times Z_2^d \times Z_2^u$ , we assign leptons as shown in Table 4 and Higgs triplets as shown in Table 5.



symmetry	$(\nu_e, e)$	$[(\nu_\mu, \mu), (\nu_\tau, \tau)]$	$e^c$	$[(\mu^c, \tau^c)]$
$D_7$	<b>1</b>	<b>2<sub>1</sub></b>	<b>1</b>	<b>2<sub>3</sub></b>
$Z_2^d$	+	+	+	+
$Z_2^u$	+	+	+	+

Table 5: Lepton assignments under  $D_7 \times Z_2^d \times Z_2^u$ .

symmetry	$\xi_1$	$\xi_{2,3}$
$D_7$	<b>1</b>	<b>2<sub>1</sub></b>
$Z_2^d$	+	+
$Z_2^u$	+	+

Table 6: Higgs triplet assignments under  $D_7 \times Z_2^d \times Z_2^u$ .

As a result, the  $(e, \mu, \tau)$  mass matrix is diagonal, coming from the Yukawa terms  $ee^c\bar{\phi}_1^0$  and  $\mu\mu^c\bar{\phi}_5^0 + \tau\tau^c\bar{\phi}_6^0$ . As for the Majorana  $(\nu_e, \nu_\mu, \nu_\tau)$  mass matrix, the allowed Yukawa terms are  $\nu_e\nu_e\xi_1^0$ ,  $(\nu_\mu\nu_\tau + \nu_\tau\nu_\mu)\xi_1^0$ , and  $\nu_e(\nu_\mu\xi_3^0 + \nu_\tau\xi_2^0)$ . The resulting mass matrix is thus of the form [11]

$$\mathcal{M}_\nu = \begin{pmatrix} a & c & d \\ c & 0 & b \\ d & b & 0 \end{pmatrix}, \quad (25)$$

which was first derived using  $Q_8$  [12], and realizes one of the conjectures of Ref. [13].

## 6 Analysis of Neutrino Mass Matrix

Rotating  $\mathcal{M}_\nu$  to the tribimaximal basis using

$$\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = U_{TB}^\dagger \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} \sqrt{2/3} & -\sqrt{1/6} & -\sqrt{1/6} \\ \sqrt{1/3} & \sqrt{1/3} & \sqrt{1/3} \\ 0 & -\sqrt{1/2} & \sqrt{1/2} \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}, \quad (26)$$

it becomes

$$\mathcal{M}_\nu^{(1,2,3)} = \begin{pmatrix} m_1 & m_6 & m_4 \\ m_6 & m_2 & m_5 \\ m_4 & m_5 & m_3 \end{pmatrix}, \quad (27)$$

where

$$m_1 = \frac{1}{3}(2a + b - 2c - 2d), \quad (28)$$

$$m_2 = \frac{1}{3}(a + 2b + 2c + 2d), \quad (29)$$

$$m_3 = -b, \quad (30)$$

$$m_4 = \frac{1}{\sqrt{3}}(-c + d), \quad (31)$$

$$m_5 = \frac{1}{\sqrt{6}}(-c + d) = \frac{m_4}{\sqrt{2}}, \quad (32)$$

$$m_6 = \frac{1}{3\sqrt{2}}(2a - 2b + c + d) = \frac{1}{2\sqrt{2}}(m_1 + 2m_2 + 3m_3). \quad (33)$$

If  $m_4 = m_5 = m_6 = 0$ , tribimaximal mixing is recovered. In particular,  $m_4 \neq 0$  or  $m_5 \neq 0$  means that  $\theta_{13} \neq 0$ . In previous studies, the special cases  $m_4 \neq 0, m_5 = m_6 = 0$  [16, 17] and  $m_5 \neq 0, m_4 = m_6 = 0$  [8, 9, 18] have been explored. The requirement from  $D_7$  that  $m_5 = m_4/\sqrt{2}$  is a new condition which will predict a special correlation between  $\theta_{13}$  and  $\theta_{23}$  as well as  $\delta_{CP}$ .

Consider the unitary matrix  $U_\epsilon$  such that

$$U_\epsilon^\dagger \mathcal{M}_\nu^{(1,2,3)} (\mathcal{M}_\nu^{(1,2,3)})^\dagger U_\epsilon = \begin{pmatrix} |m'_1|^2 & 0 & 0 \\ 0 & |m'_2|^2 & 0 \\ 0 & 0 & |m'_3|^2 \end{pmatrix}, \quad (34)$$

then  $U'_{\alpha i} = U_{TB} U_\epsilon$  is the lepton mixing matrix up to phases. Let  $U_\epsilon$  be approximately given by

$$U_\epsilon = \begin{pmatrix} 1 & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & 1 & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & 1 \end{pmatrix}, \quad (35)$$

then for  $|m'_1|^2 \simeq |m_1|^2$ , we have

$$\epsilon_{21} \simeq \frac{-(m_6 m_1^* + m_2 m_6^*)}{|m_2|^2 - |m_1|^2}. \quad (36)$$

In addition, since the effective neutrino mass  $m_{ee}$  in neutrinoless double beta decay is given by

$$m_{ee} = |a| = |m_1 + m_2 + m_3|, \quad (37)$$

whereas

$$m_3 = \frac{1}{3}(2\sqrt{2}m_6 - m_1 - 2m_2), \quad (38)$$

we have the relationship

$$|m_3|^2 - m_{ee}^2 = \frac{1}{3}(|m_2|^2 - |m_1|^2)[1 + 4\sqrt{2}Re(\epsilon_{21})]. \quad (39)$$

Since  $|m_2|^2 - |m_1|^2 \simeq \Delta m_{21}^2$  is very small, this model predicts  $m_{ee} = |m_3|$  to a very good approximation. The structure of Eq. (38) also shows that an inverted ordering of neutrino masses is expected, although the quasidegenerate limit is also possible in which case either inverted or normal ordering may occur. In the following we focus on the inverted case, i.e.  $|m_3| < |m_1| < |m_2|$ .

For  $m_4 \neq 0$ ,  $\nu_3$  is rotated to  $\nu'_3$  according to

$$\epsilon_{13} \simeq \frac{m_1 m_4^* + m_4 m_3^*}{|m_3|^2 - |m_1|^2}, \quad \epsilon_{23} \simeq \frac{m_2 m_4^* + m_4 m_3^*}{\sqrt{2}(|m_3|^2 - |m_1|^2)}. \quad (40)$$

As a result,

$$U'_{e3} \simeq \sqrt{\frac{2}{3}}\epsilon_{13} + \sqrt{\frac{1}{3}}\epsilon_{23} \simeq \frac{-m_4(m_1 + 2m_2)^* + m_4^*(2m_1 + m_2)}{\sqrt{6}(|m_1|^2 - |m_3|^2)}, \quad (41)$$

$$U'_{\mu 3} \simeq -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}}\epsilon_{13} + \frac{1}{\sqrt{3}}\epsilon_{23} \simeq -\frac{1}{\sqrt{2}} - \frac{(m_1 - m_2)m_4^*}{\sqrt{6}(|m_1|^2 - |m_3|^2)}, \quad (42)$$

$$U'_{\tau 3} \simeq \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}}\epsilon_{13} + \frac{1}{\sqrt{3}}\epsilon_{23} \simeq \frac{1}{\sqrt{2}} - \frac{(m_1 - m_2)m_4^*}{\sqrt{6}(|m_1|^2 - |m_3|^2)}, \quad (43)$$

If all parameters are real, then for  $U'_{e3} = 0.16$ ,  $\sin^2 2\theta_{23}$  would be 0.80, which is ruled out by present data, i.e.  $\sin^2 2\theta_{23} > 0.92$ . However, a fit may be obtained for complex values.

We go back to Eq. (25) and observe that  $a, c, d$  may be chosen real, so only  $b$  is complex. This means that  $m_4$  is real as well as  $2m_1 - m_2$ , and for  $m_6 = 0$ ,  $m_3 = -(m_1 + 2m_2)/3$ . Writing  $m_{1,2}$  as  $m_{1,2} e^{i\phi_{1,2}}$  with  $m_2 \simeq m_1$  and  $\sin \phi_2 = 2 \sin \phi_1$ , we obtain

$$U'_{e3} \simeq \frac{m_1 m_4}{\sqrt{6}\Delta m_{32}^2} [-\cos \phi_1 + \cos \phi_2 - 9i \sin \phi_1], \quad (44)$$

$$U'_{\mu 3} \simeq -\frac{1}{\sqrt{2}} + \frac{m_1 m_4}{\sqrt{6} \Delta m_{32}^2} [\cos \phi_1 - \cos \phi_2 - i \sin \phi_1], \quad (45)$$

$$U'_{\tau 3} \simeq \frac{1}{\sqrt{2}} + \frac{m_1 m_4}{\sqrt{6} \Delta m_{32}^2} [\cos \phi_1 - \cos \phi_2 - i \sin \phi_1], \quad (46)$$

where  $\cos \phi_2 = \pm \sqrt{1 - 4 \sin^2 \phi_1}$ . We then have

$$\sin^2 \theta_{13} = \frac{|U'_{e3}|^2}{1 + |\epsilon_{13}|^2 + |\epsilon_{23}|^2}, \quad \tan^2 \theta_{23} = \frac{|U'_{\mu 3}|^2}{|U'_{\tau 3}|^2}. \quad (47)$$

Since

$$|m_3| \simeq \frac{\sqrt{\Delta m_{32}^2} \sqrt{5 + 4 \cos(\phi_2 - \phi_1)}}{2 \sqrt{1 - \cos(\phi_2 - \phi_1)}}, \quad (48)$$

the above equations relate  $|m_3| = m_{ee}$  with  $\theta_{13}$  and  $\theta_{23}$ . If we fix  $\theta_{13}$ , we then obtain  $|m_3|$  as a function of  $\theta_{23}$ . We plot in Fig. 2 our model predictions for  $|m_{1,2}|$  and  $|m_3| = m_{ee}$  versus  $\sin^2 2\theta_{23}$ . The other data points are taken to be their experimental central values.

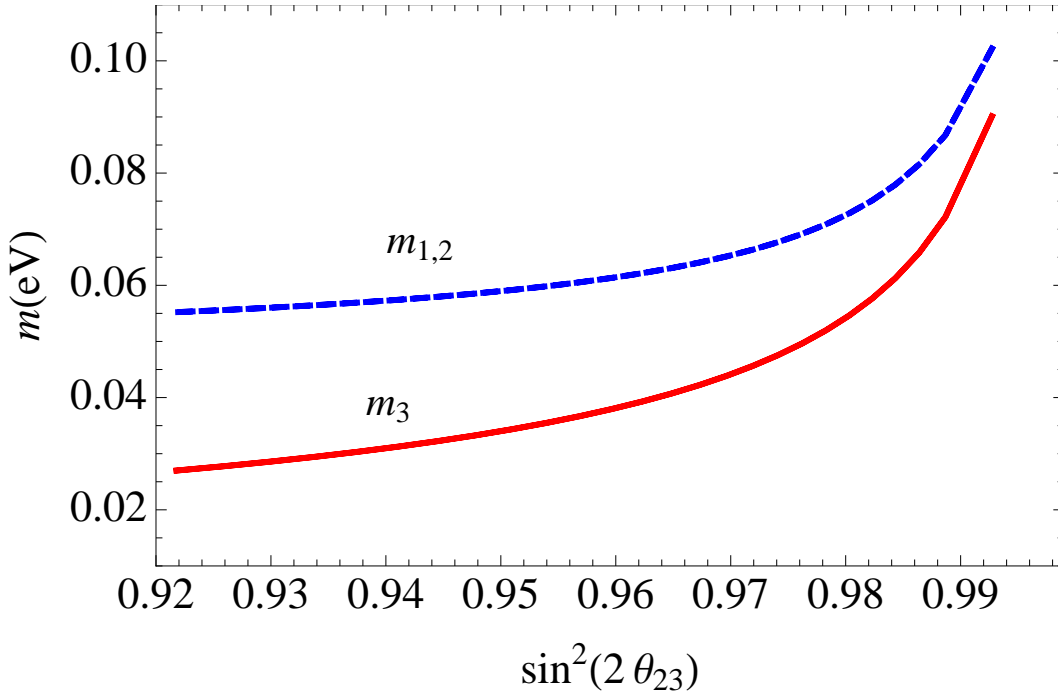


Figure 2: Neutrino masses  $m_{1,2}$  and  $m_3 = m_{ee}$  versus  $\sin^2 2\theta_{23}$ .

If we rotate  $\mathcal{M}_\nu^{1,2,3}(\mathcal{M}_\nu^{1,2,3})^\dagger$  by

$$U'_\epsilon = \begin{pmatrix} 1 & 0 & \epsilon_{13} \\ 0 & 1 & \epsilon_{23} \\ -\epsilon_{13}^* & -\epsilon_{23}^* & 1 \end{pmatrix}, \quad (49)$$

we obtain the  $2 \times 2$  mass-squared matrix spanning  $\nu'_{1,2}$ . This differs from the  $2 \times 2$  submatrix in the tribimaximal basis by terms quadratic in  $m_4$  which are important in obtaining the correct  $\Delta m_{21}^2$  and Eq. (36) becomes modified. However, we can adjust  $|m_2|$  versus  $|m_1|$  as well as  $m_6$  to fit the data. These adjustments will have negligible effects on  $|m_3|$ .

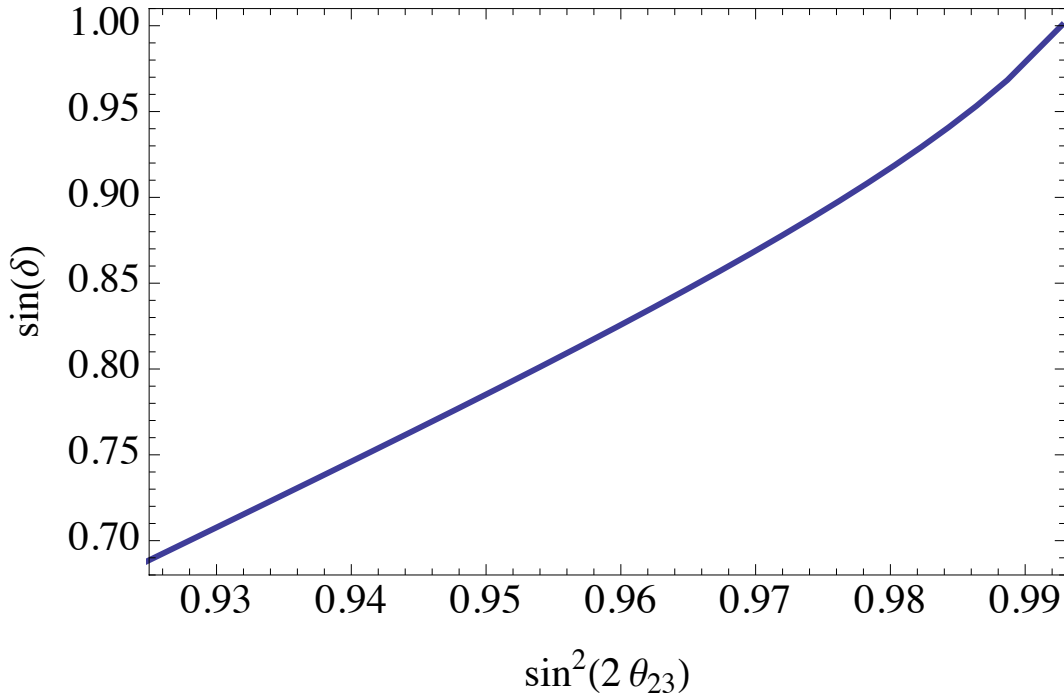


Figure 3: The  $CP$  violating parameter  $|\sin \delta_{CP}|$  versus  $\sin^2 2\theta_{23}$ .

We plot in Fig. 3 our model prediction for  $|\sin \delta_{CP}|$  versus  $\sin^2 2\theta_{23}$ . To obtain  $\sin \delta_{CP}$ , we use

$$U'_{e2} \simeq \frac{1}{\sqrt{3}}, \quad U'_{\mu 2} \simeq \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}}\epsilon_{23}^*, \quad J = \text{Im}(U'_{e2}U'_{\mu 3}U'_{\mu 2}{}^*U'_{e3}{}^*), \quad (50)$$

from which we find (using  $U'_{\mu 2} = |U'_{\mu 2}|e^{i\theta_{\mu 2}}$ , etc.)

$$\sqrt{\frac{2}{3}} \cos \theta_{23} \sin \delta \simeq |U'_{\mu 2}| \sin(\theta_{\mu 3} - \theta_{\mu 2} - \theta_{e3}). \quad (51)$$

## 7 Concluding Remarks

We have studied a specific pattern for both quark and lepton mass matrices. In both cases, one mass matrix is diagonal ( $\mathcal{M}_u$  and  $\mathcal{M}_e$ ), whereas the other has two zeros ( $\mathcal{M}_d$  and  $\mathcal{M}_\nu$ ). In the case of  $\mathcal{M}_\nu$ , the assumption that it is Majorana corresponds to one of the conjectures of Ref. [13], whereas the Dirac mass matrix  $\mathcal{M}_d$  requires further restrictions to make it predictive, as first proposed in Ref. [10] using the non-Abelian discrete symmetry  $D_7$ . The conjectured form of  $\mathcal{M}_\nu$  was first derived [12] using  $Q_8$ , but it may also be obtained [11] using  $D_5$  or  $D_7$ . Here we consider  $D_7$  as the unifying symmetry for both quarks and leptons.

The  $CP$  violating parameter  $J$  in the quark sector is constrained in this model by  $m_d, m_s, m_b, |V_{us}|, |V_{ub}|, |V_{cb}|$ . Within one standard deviation of all six measurements, we obtain  $J$  in agreement with data. In the neutrino sector, we obtain  $|m_{1,2}|$  as well as  $|m_3| = m_{ee}$  as functions of  $\sin^2 2\theta_{23}$  and also predict  $\sin \delta_{CP}$  as a function of  $\sin^2 2\theta_{23}$ .

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